Section 11.2

Power Series

While we could certainly view power series as an extension of Taylor polynomials, in particular setting $g(x) = \lim_{n\to\infty} p_n(x)$ where p_n is the nth-order Taylor polynomial for some function f, we will instead view power series as a generalization of the series we have already studied.

Definition 11.2.1. Power Series

A *power series* is a function of the general form

$$f(x) = \sum_{k=0}^{\infty} c_k(x-a)^k$$

where a and c_k are real numbers and x is an independent variable. The c_k's are the coefficients of the power series and a is the *center*. The set of values of x for which the series converges is its *interval of convergence*. The *radius of convergence*, denoted R, is the distance from the center of the series to the boundary of the interval of convergence.

Example 11.2.2. Find a closed form for the power series $f(x) = \sum_{k=0}^{\infty} x^k$. Find the interval of convergence for f.

Theorem 11.2.3. Convergence of Power Series

A power series $\sum_{k=0}^{\infty} c_k(x-a)^k$ centered at a converges in one of three ways:

1. The series converges absolutely for all x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.
2. There is a real number $R > 0$ such that the series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$, in which case the radius of convergence is R.
3. The series converges only at a, in which case the radius of convergence is $R = 0$.

Example 11.2.4. Find the radius and interval of convergence:

1. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

2. $\sum_{k=1}^{\infty} k! x^k$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}$

A power series defines a function on its interval of convergence. We can then combine power series with other function to make new functions.

Theorem 11.2.5. Combining Power Series

Suppose the power series $\sum c_k x^k$ and $\sum d_k x^k$ converge to $f(x)$ and $g(x)$ on an interval I.

1. **Sum and difference.** The power series $\sum (c_k \pm d_k) x^k$ converges to $f(x) \pm g(x)$ on I.
2. **Multiplication by a power function.** Suppose m is an integer such that $k + m \geq 0$ for all k in the index of the power series $x^m \sum c_k x^k = \sum c_k x^{k+m}$. This series converges to $x^m f(x)$ for all $x \neq 0$ in I. When $x = 0$, the series converges to $\lim_{x \to \infty} x^m f(x)$.
3. **Composition.** If $h(x) = bx^m$ where m is a positive integer and b is a nonzero real number, the power series $\sum c_k (h(x))^k$ converges to the composition $f(h(x))$ for all x such that $h(x)$ is in I.

Example 11.2.6. Find the power series and radius of convergence for the following functions.

1. $\dfrac{x^5}{1-x}$

2. $\dfrac{1}{1-2x}$

Differentiating and Integrating Power Series

One expects integration and differentiation of power series to behave in the same manner as polynomials. This is true, provided we work over the interval of convergence.

Theorem 11.2.7. Let $f(x) = \sum c_k(x-a)^k$ where equality holds on the interval convergence of the RHS.

1. Then f is differentiable for $|x-a| < R$. Further

$$f'(x) = \sum kc_k(x-a)^{k-1}$$

for $|x-a| < R$.

2. The indefinite integral of f is found by integrating the power series of f term by term:

$$\int f(x)\,dx = \sum c_k \frac{(x-a)^{k+1}}{k+1} + C$$

for $|x-a| < R$.

Example 11.2.8. Find the power series of $\ln|1-x|$ and $\arctan x$.

Example 11.2.9. Evaluate the series $\displaystyle\sum_{k=0}^{\infty} k\left(\frac{1}{3}\right)^k$